12 - Predicate logic - solutions

Problem 1.
(a) $\exists x \in \mathbb{R} : x^2 > x$ is true: $x = 2$ is a real number and $2^2 = 4 > 2$.
(b) $\exists x \in \mathbb{R} : x^2 = -1$ is false: x^2 cannot be negative if x is a real number.
(c) $\exists x \in \mathbb{R} : x^2 + 2 > 1$ is true: $x = 1$ is a real number and makes this true: $x^2 + 2 = 1^2 + 2 = 3 > 1$.
(d) $\forall x \in \mathbb{N} : (x^2 \neq x) \vee (x = 0) \vee (x = 1)$ is true: the last two cases is true if $x = 0$ or if $x = 1$; otherwise x is a natural number ≥ 2 and so $x^2 > x$ which also means that $x^2 \neq x$.
(e) $\exists x \in \mathbb{N} : n^2 \equiv 3 \pmod 4$ is false: of the four cases to check ($n \equiv 0, 1, 2, \text{ or } 3 \pmod 4$), each case results in $n^2 \equiv 0 \text{ or } 1 \pmod 4$ so $n^2 \equiv 3 \pmod 4$ cannot occur.

Problem 2.
The proposition $\forall x \in \mathbb{R} : x^2 \geq x$ is false if we take $x = 1/2$ since $(1/2)^2 = 1/4$ which is less than $1/2$.

Problem 3.
Let $C(x) =$"x has a cat", $D(x) =$"x has a dog", $F(x) =$"x has a ferret.", $S = \{\text{students in your class}\}$. Formalize:

Part (a). A student in your class has a cat, a dog, and a ferret:
In progressively more detail, this sentence reads as:

- "There exists a student in your class who has a cat, a dog, and a ferret"
- "There exists a student x in your class such that x has a cat, a dog, and a ferret"
- "There exists a student x in your class such that x has a cat, and x has a dog, and x has a ferret"

Answer: $\exists x \in S : C(x) \wedge D(x) \wedge F(x)$

Part (b). All students in your class have a cat, a dog, or a ferret.
In progressively more detail, this sentence reads as:

- "For all students x in your class, x has a cat, a dog, or a ferret"
- "For all students x in your class, x has a cat, or x has a dog, or x has a ferret"

Answer: $\forall x \in S : C(x) \vee D(x) \vee F(x)$

Part (c). Some student in your class has a cat and a ferret but not a dog.

Answer: $\exists x \in S : C(x) \wedge F(x) \wedge \neg D(x)$

Part (d). No student in this class has a cat, a dog, and a ferret.

No student means exactly zero students. The opposite of zero students is at least one students. Therefore:

"No student in this class has a cat, a dog, and a ferret."
≡ "It's not the case that at least one student in this class has a cat, a dog, and a ferret."
≡ $\neg(\exists x \in S : C(x) \wedge D(x) \wedge F(x))$

You can leave your answer like this, or you can also apply de Morgan's law to simplify this to:

$\forall x \in S : \neg C(x) \vee \neg D(x) \vee \neg F(x)$

Part (e). For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has (at least) one of these animals.

This is slightly tricky as well. We cannot quantify over the animals via $\forall$ directly. But $\forall$ is basically a supersized version of AND, so we rewrite the sentence to:

"There is a student in your class who has a cat AND There is a student in your class who has a dog AND There is a student in your class who has a ferret. "

Answer: $(\exists x \in S : C(x)) \wedge (\exists x \in S : D(x)) \wedge (\exists x \in S : F(x))$